Thermalization and Plasma Instabilities

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Strong and Electroweak Matter 2006

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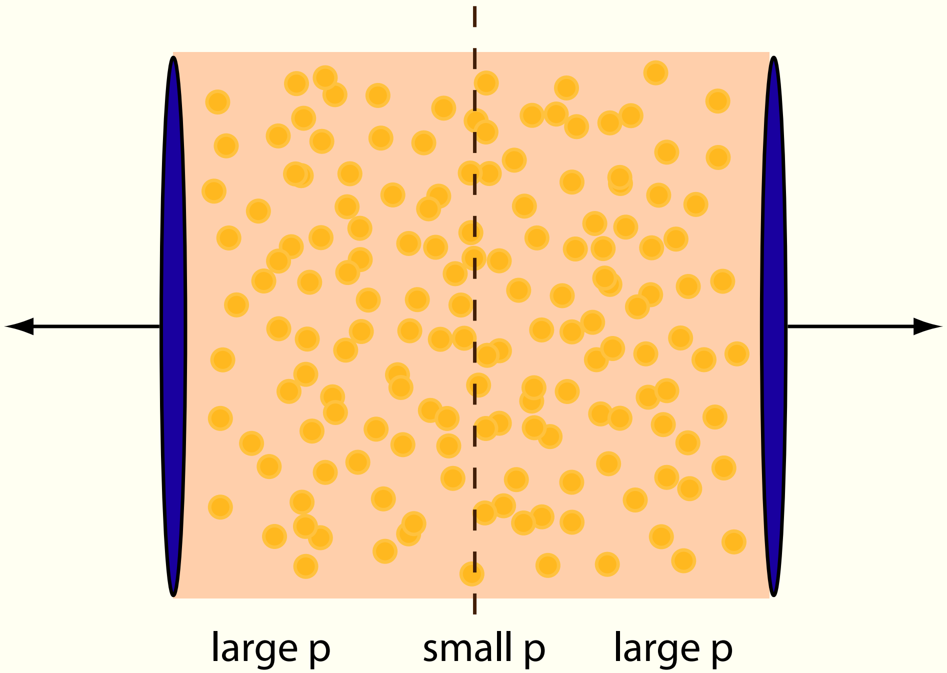
Motivation

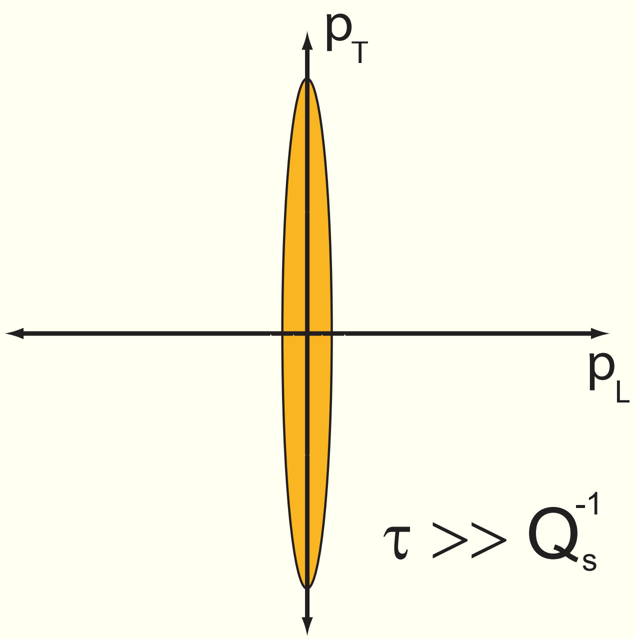
- We would like to have a first principles derivation of the mechanisms and time scales necessary for the isotropization and equilibration of a quark-gluon plasma.
- In addition to equilibration via 2-2 elastic scattering (super slow) one needs to include inelastic processes, e.g. bremstrahlung 2-3 (and 3-2) processes, and the effect of background fields.
- In equilibrium the background field (soft modes) only serves to screen the interaction (Debye screening). However, in a non-equilibrium setting the background field can have non-trivial dynamics.
- Consider, for example, a spatially homogeneous plasma which has been initialized such that it has a “temperature” anisotropy.
- In such an anisotropic plasmas new collective modes corresponding to *electro-/chromodynamic instabilities* appear.

Why anisotropic distribution functions?

Because of the natural expansion of the system the gluon distribution functions created during relativistic heavy ion collisions are *generically* locally anisotropic in momentum space.

$$
\begin{aligned}
< p_T > \quad &\sim \quad Q_s \quad && \text{(nuclear saturation scale)} \\
< p_L > \quad &\sim \quad 1/\tau \quad && \text{(free streaming)}
\end{aligned}
$$





Current Filamentation

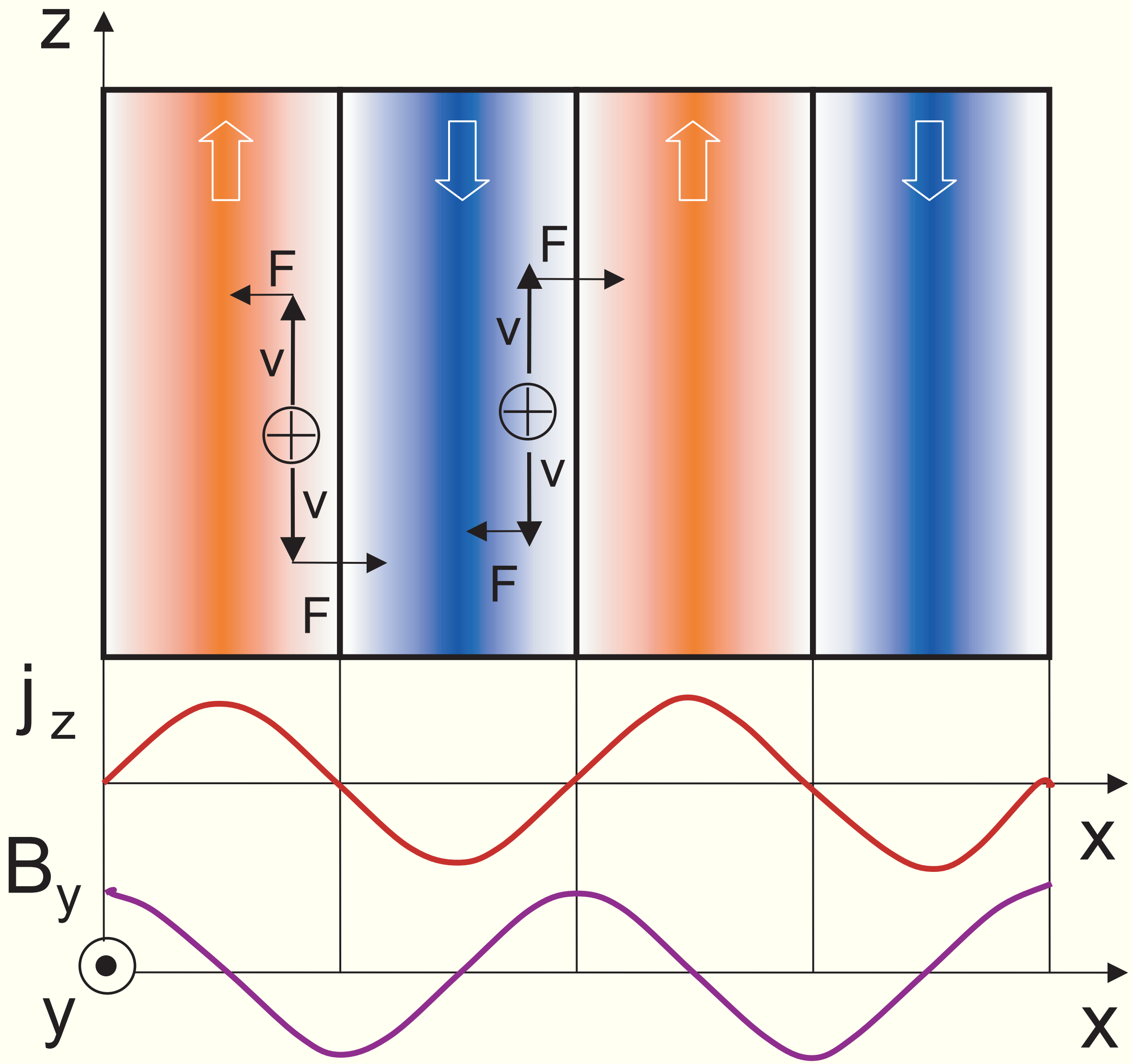
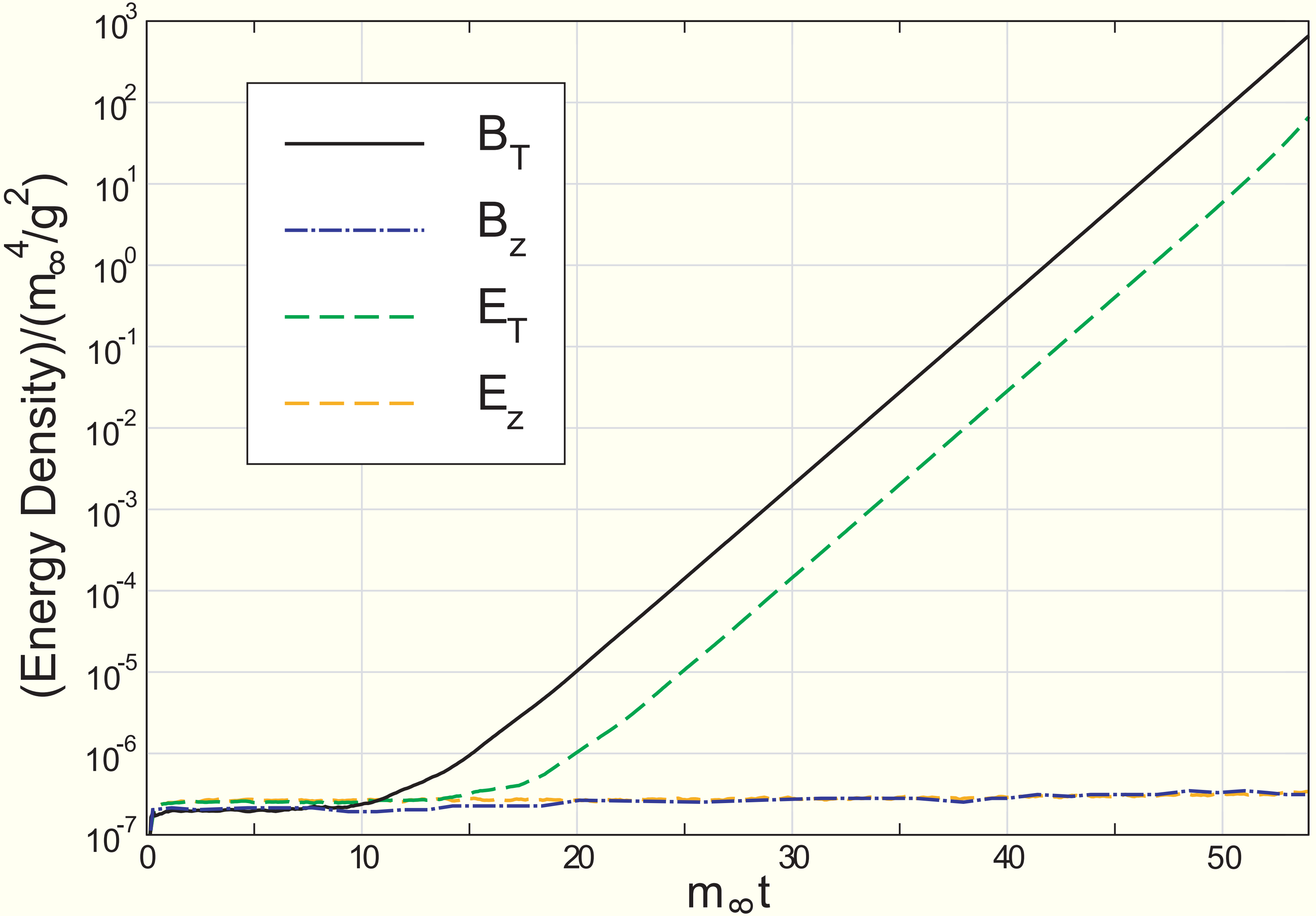


Figure courtesy S. Mrówczyński

Abelian Plasma



Collective Modes of an Isotropic QGP

The isotropic hard-thermal-loop (HTL) gluon propagator is given by

$$\Delta^{ij} = (k^2 - \omega^2 + \Pi_T)^{-1}(\delta_{ij} - k^i k^j/k^2) - \frac{k^2}{\omega^2}(k^2 - \Pi_L)^{-1} k^i k^j/k^2$$

with

$$\begin{aligned}
\Pi_T(\omega,k) &= \frac{m_D^2}{2}\frac{\omega^2}{k^2}\left[1 - \frac{\omega^2 - k^2}{2\omega k}\log\frac{\omega+k}{\omega-k}\right] , \\
\Pi_L(\omega,k) &= m_D^2\left[\frac{\omega}{2k}\log\frac{\omega+k}{\omega-k} - 1\right] ,
\end{aligned}$$

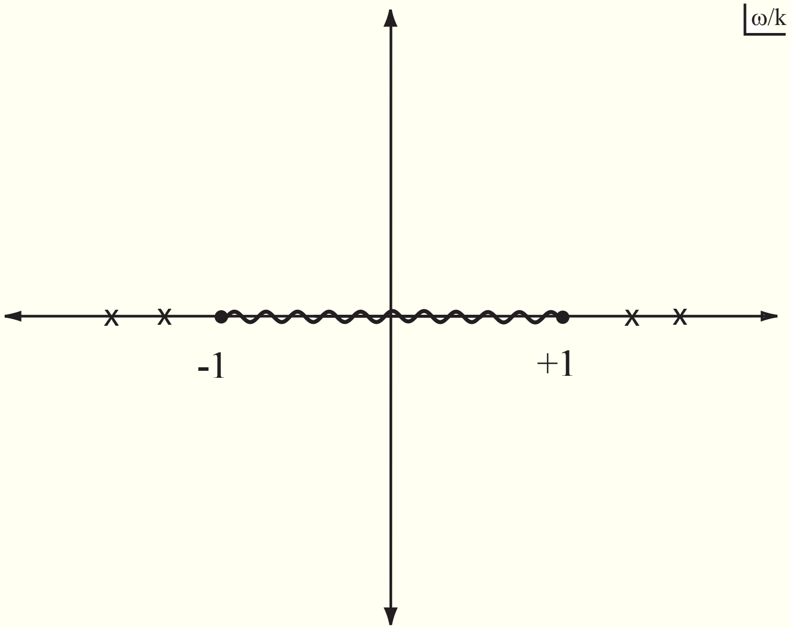
and $m_D \propto gT$.

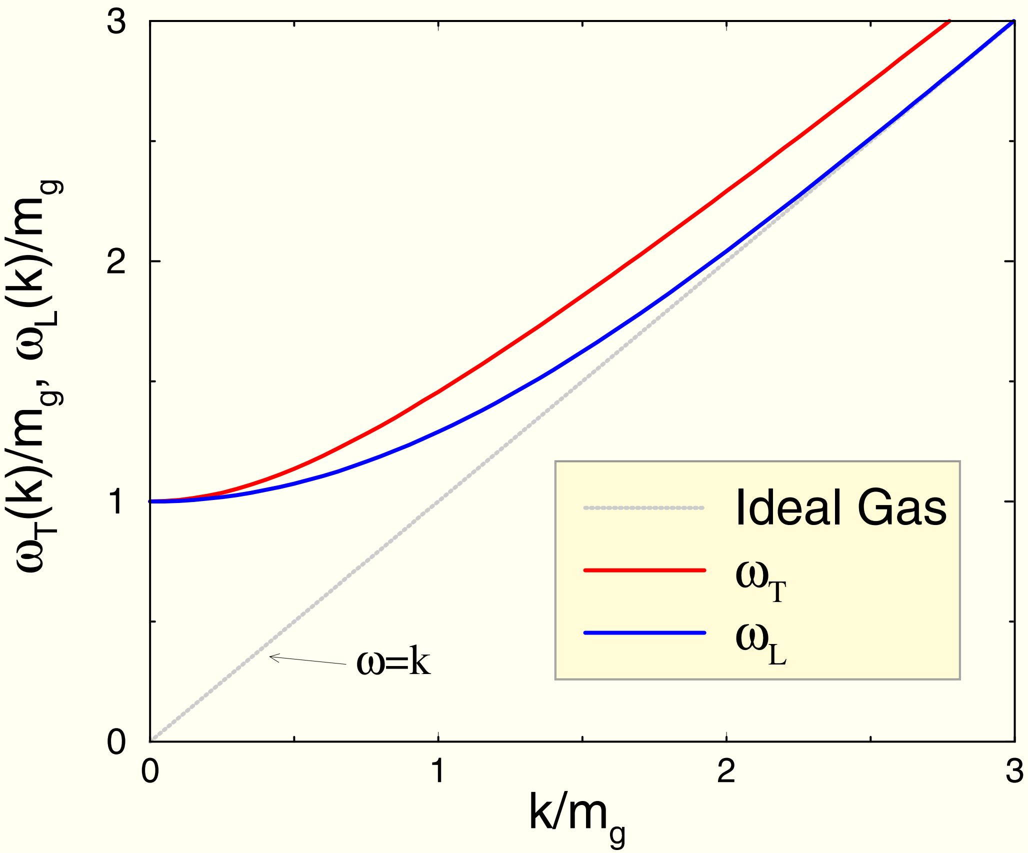
$$\begin{aligned}
\lim_{\omega\to 0}\Pi_L(\omega,k) &= m_D^2 \quad \text{electric screening} \\
\lim_{\omega\to 0}\Pi_T(\omega,k) &= 0 \quad\quad \textbf{no} \text{ magnetic screening}
\end{aligned}$$

Collective Modes of an Isotropic QGP

In the isotropic case the only poles are at real timelike ($\omega > k$) momentum. In order to determine the dispersion relations for these excitations we can then explicitly look for the poles in the propagator.

$$
\begin{aligned}
0 &= k^2 - \omega_T^2 + \Pi_T(\omega_T, k) \\
0 &= k^2 - \Pi_L(\omega_L, k)
\end{aligned}
$$





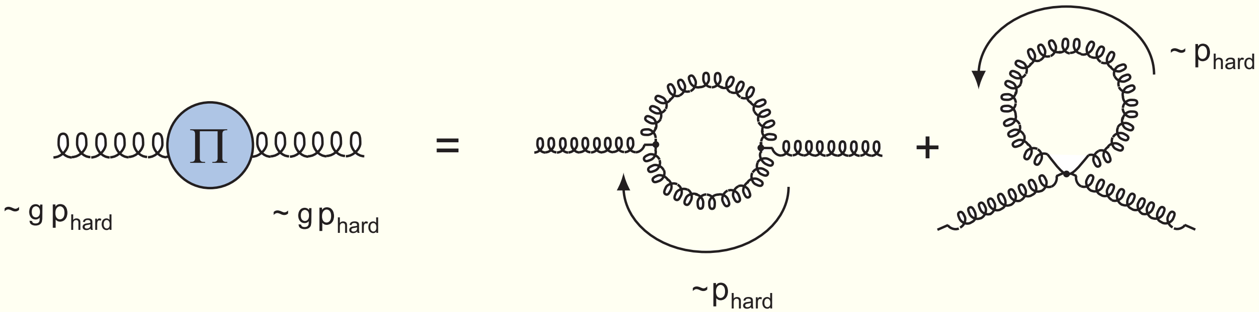
Anisotropic Gluon Polarization Tensor

In order to determine the HL gluon polarization we can use either linearized three-dimensional kinetic theory (Boltzmann-Vlasov eq)

$$[v \cdot D_X, \delta n(p, X)] + g v_\mu F^{\mu\nu}(X) \partial_\nu^{(p)} n(\mathbf{p}) = 0$$

$$D_\mu F^{\mu\nu} = J^\nu = g \int_p v^\nu \delta n(p, X)$$

or diagrammatically



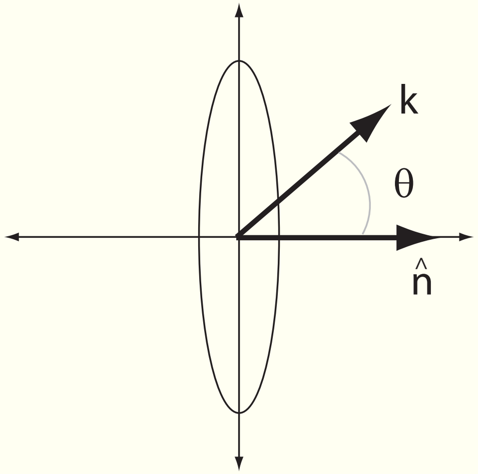
In both cases the result for the retarded self-energy is

$$\Pi_{ab}^{ij}(K) = -g^2 \delta_{ab} \int_{\mathbf{p}} v^i \partial_l f(\mathbf{p}) \left(\delta^{jl} - \frac{v^j k^l}{K \cdot V + i\epsilon} \right)$$

The nature of the anisotropy

We assume that the anisotropic distribution function can be obtained from an arbitrary isotropic distribution function by a change of its argument.

$$f(p^2) \to f(p^2 + \xi(p \cdot n)^2)$$



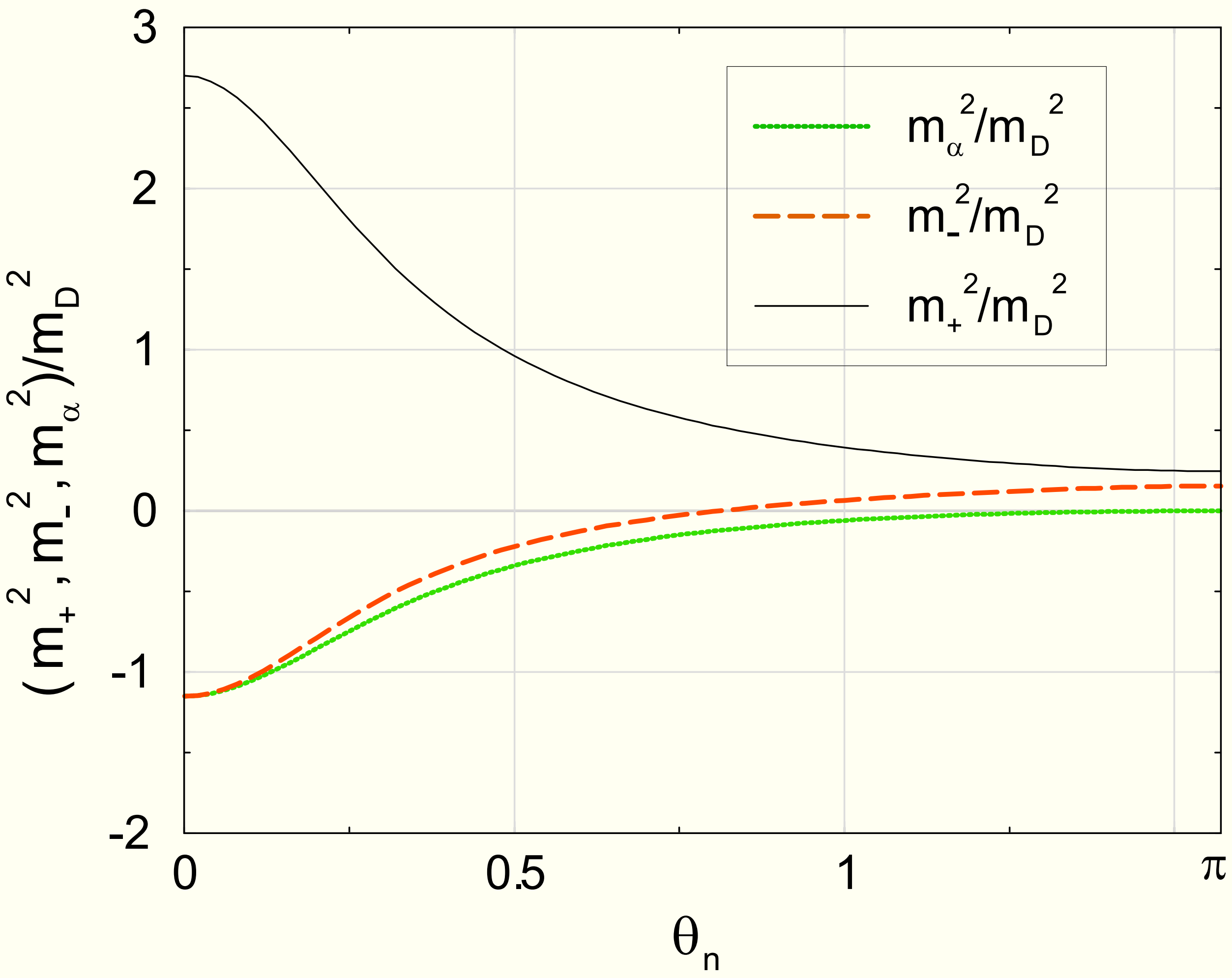
The polarization tensor can then be written as

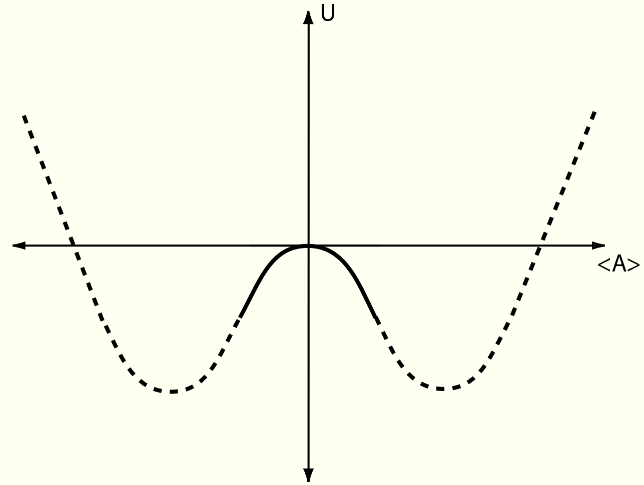
$$\Pi^{ij}(K) = m_D^2 \int \frac{d\Omega}{4\pi} v^i \frac{v^l + \xi(v \cdot n) n^l}{(1 + \xi(v \cdot n)^2)^2} \left(\delta^{jl} - \frac{v^j k^l}{K \cdot V + i\epsilon} \right)$$

where m_D is the *isotropic* Debye mass

$$m_D^2 = -\frac{g^2}{2\pi^2} \int_0^\infty dp\, p^2 \frac{df(p^2)}{dp}$$

New Mass Scales – $\xi > 0$

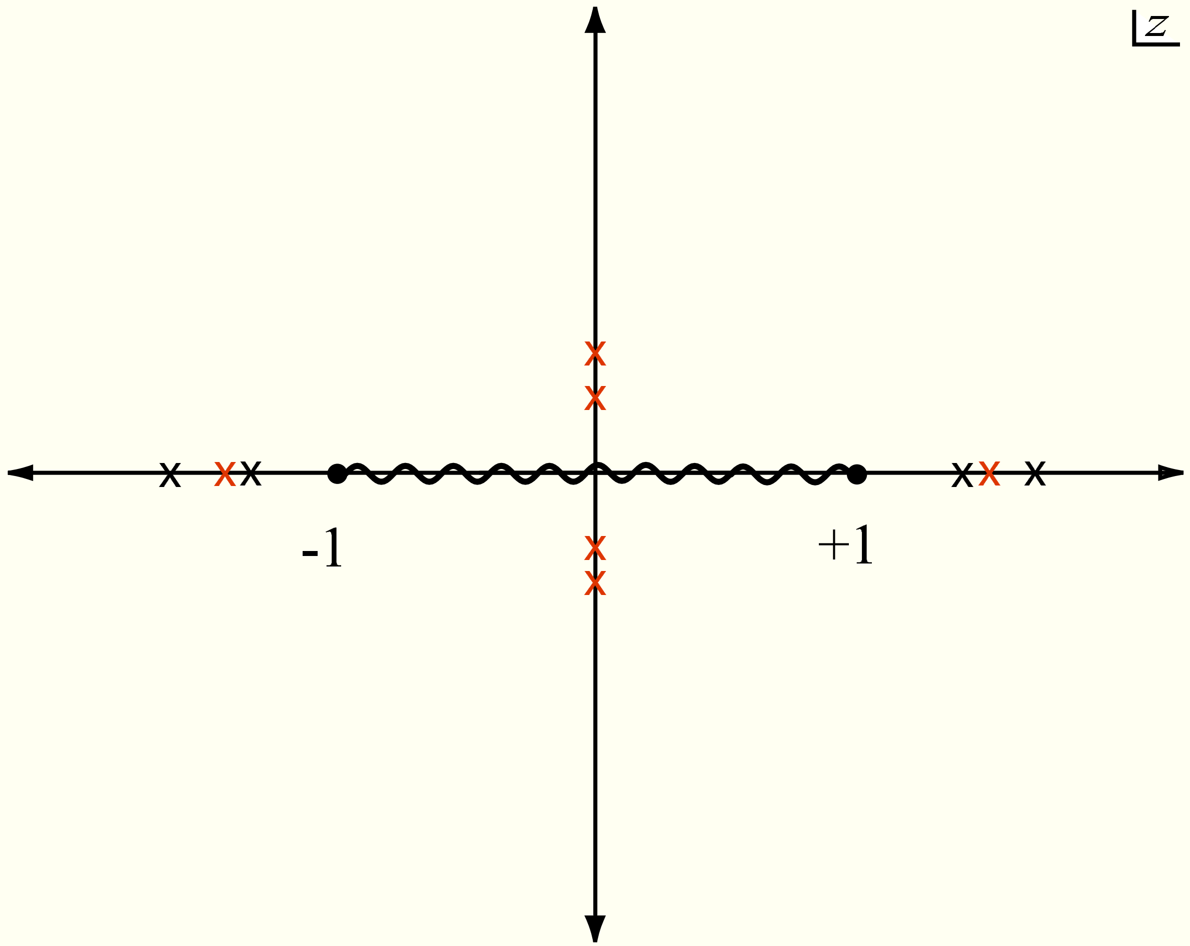




Sketch of the effective potential of an unstable mode.

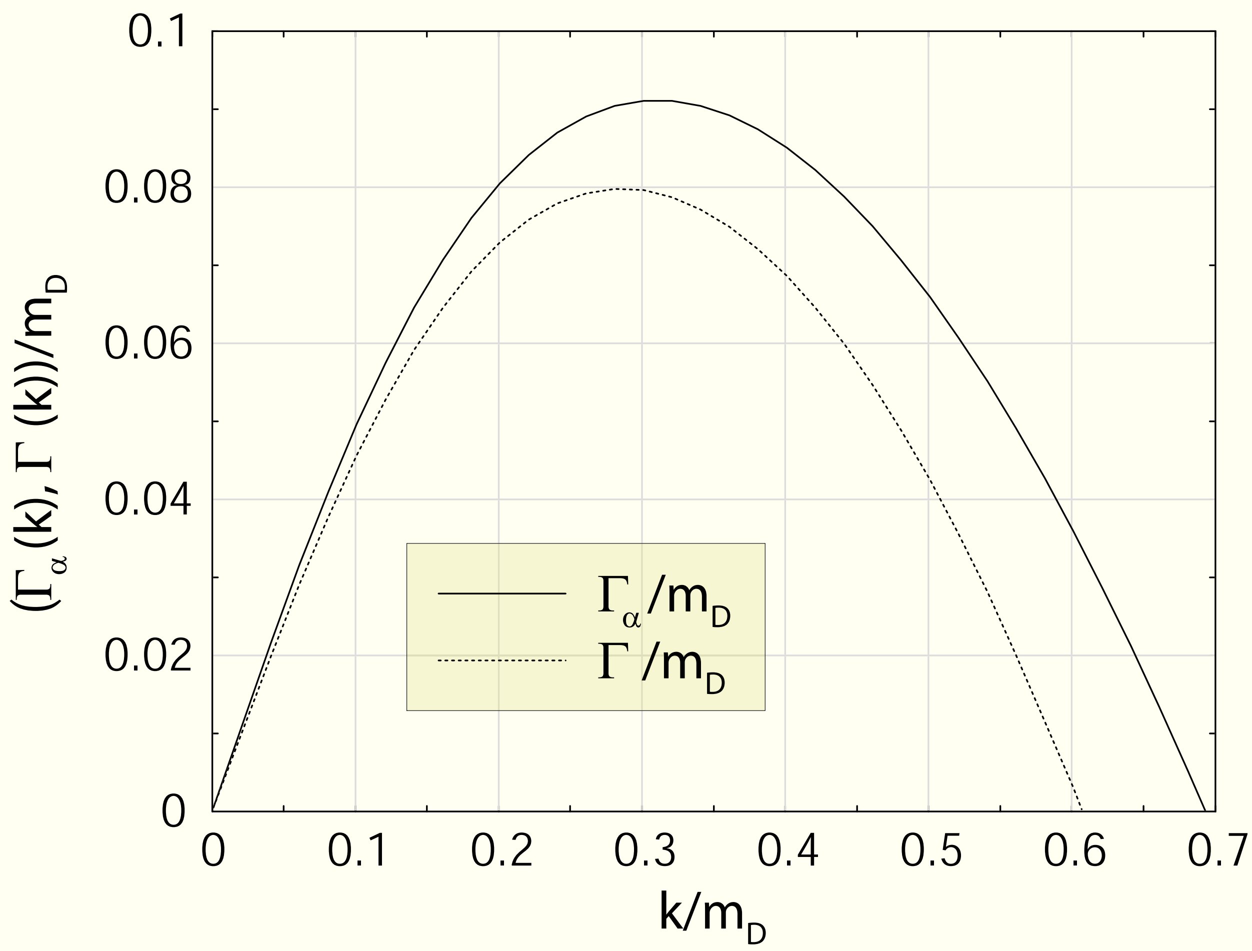
Angular dependence of m_α^2, m_+^2, and m_-^2 at fixed $\xi = 10$.

Anisotropic Collective Modes ($\xi > 0$)



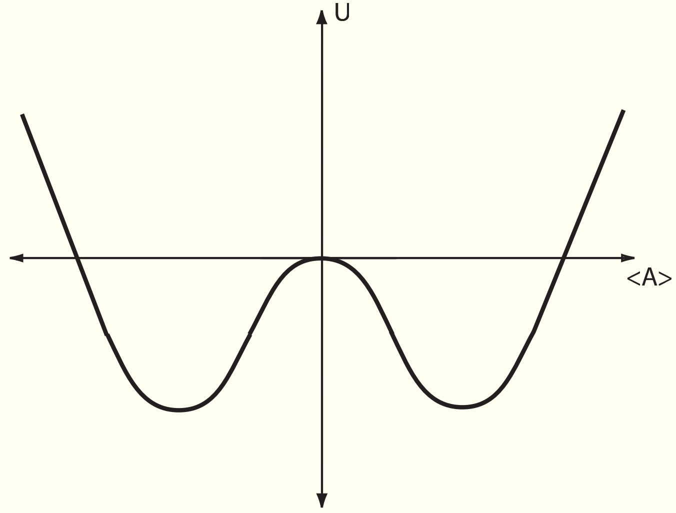
Anisotropic poles ($\xi > 0$).

Unstable Modes – $\xi > 0$



$\Gamma_\alpha(k)$ and $\Gamma_-(k)$ as a function of k for $\xi = 10$ and $\theta_n = \pi/8$.

Anisotropic HL Effective Action



Using the requirement of gauge invariance it is possible to determine all n-point functions.

$$\begin{aligned} S_{\mathrm{HL}} &= -\frac{g^2}{2}\int_x\int_{\mathbf{p}} f(\mathbf{p})\, F^a_{\mu\nu}(x)\left(\frac{p^\nu p^\rho}{(p\cdot D)^2}\right)_{ab} F_\rho{}^{b\,\mu}(x) \\ &= -\frac{g^2}{2}\int_x\int_{\mathbf{p}} f(\mathbf{p})\, W^\mu(x,\hat{\mathbf{p}})W_\mu(x,\hat{\mathbf{p}}) \end{aligned}$$

For example, from this we can obtain the anisotropic 3-gluon vertex

$$\Gamma^{\mu\nu\lambda}(k,q,r) = \frac{g^2}{2}\int_{\mathbf{p}} \frac{\partial f(\mathbf{p})}{\partial p^\beta}\,\hat{p}^\mu\hat{p}^\nu\hat{p}^\lambda\left(\frac{r^\beta}{\hat{p}\cdot q\;\hat{p}\cdot r} - \frac{k^\beta}{\hat{p}\cdot k\;\hat{p}\cdot q}\right)$$

Real-Time Lattice Simulation

Numerically solve the equations of motion resulting from the HL effective action on a space + velocity lattice.

$$j^\mu[A] = -g^2 \int_{\mathbf{p}} \frac{1}{2|\mathbf{p}|}\, p^\mu \, \frac{\partial f(\mathbf{p})}{\partial p^\beta} W^\beta(x;\mathbf{v})$$

with

$$[v \cdot D(A)] W_\beta(x;\mathbf{v}) = F_{\beta\gamma}(A) v^\gamma$$

and $v^\mu = p^\mu/|\mathbf{p}| = (1, \mathbf{v})$.

This has to be solved with

$$D_\mu(A) F^{\mu\nu} = j^\nu$$

where $\nu = 0$ is the Gauss law constraint.

$\vec{\mathrm{v}}$-discretized equations of motion

Recall,

$$j^\nu[A] = -g^2 \int_{\mathbf{p}} \frac{1}{2|\mathbf{p}|}\, p^\nu \,\frac{\partial f(\mathbf{p})}{\partial p^\beta} W^\beta(x;\mathbf{v})$$

A closed set of gauge-covariant equations is obtained when the angular integral over $\hat{\mathbf{p}}$ is discretized.

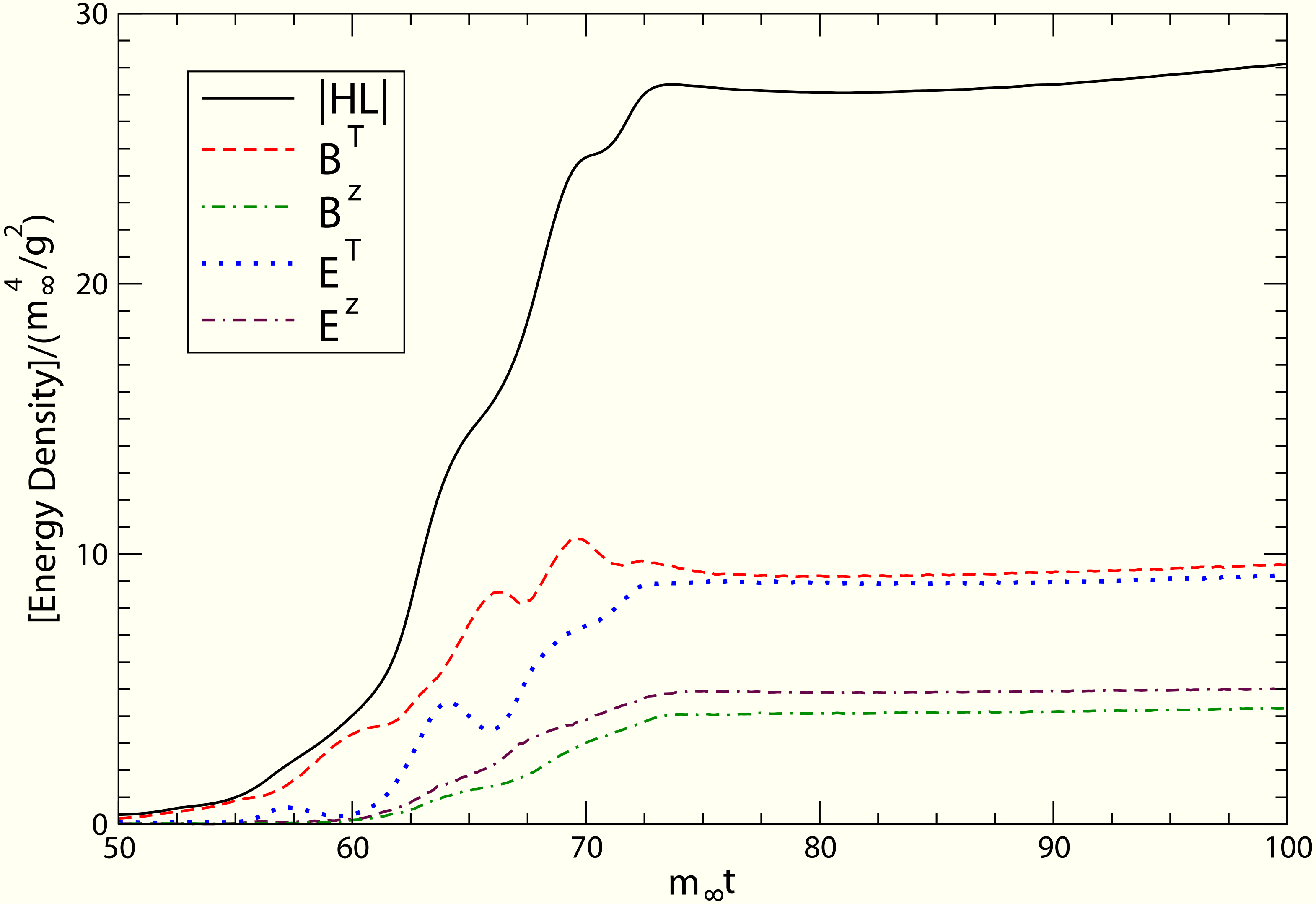
The full HL dynamics is then approximated by the following set of equations

$$[v \cdot D(A)]\mathcal{W}_{\mathbf{v}} = (a_{\mathbf{v}} F^{0\mu} + b_{\mathbf{v}} F^{z\mu}) v_\mu$$

$$D_\mu(A) F^{\mu\nu} = j^\nu = \frac{1}{\mathcal{N}} \sum_{\mathbf{v}} v^\nu \mathcal{W}_{\mathbf{v}}$$

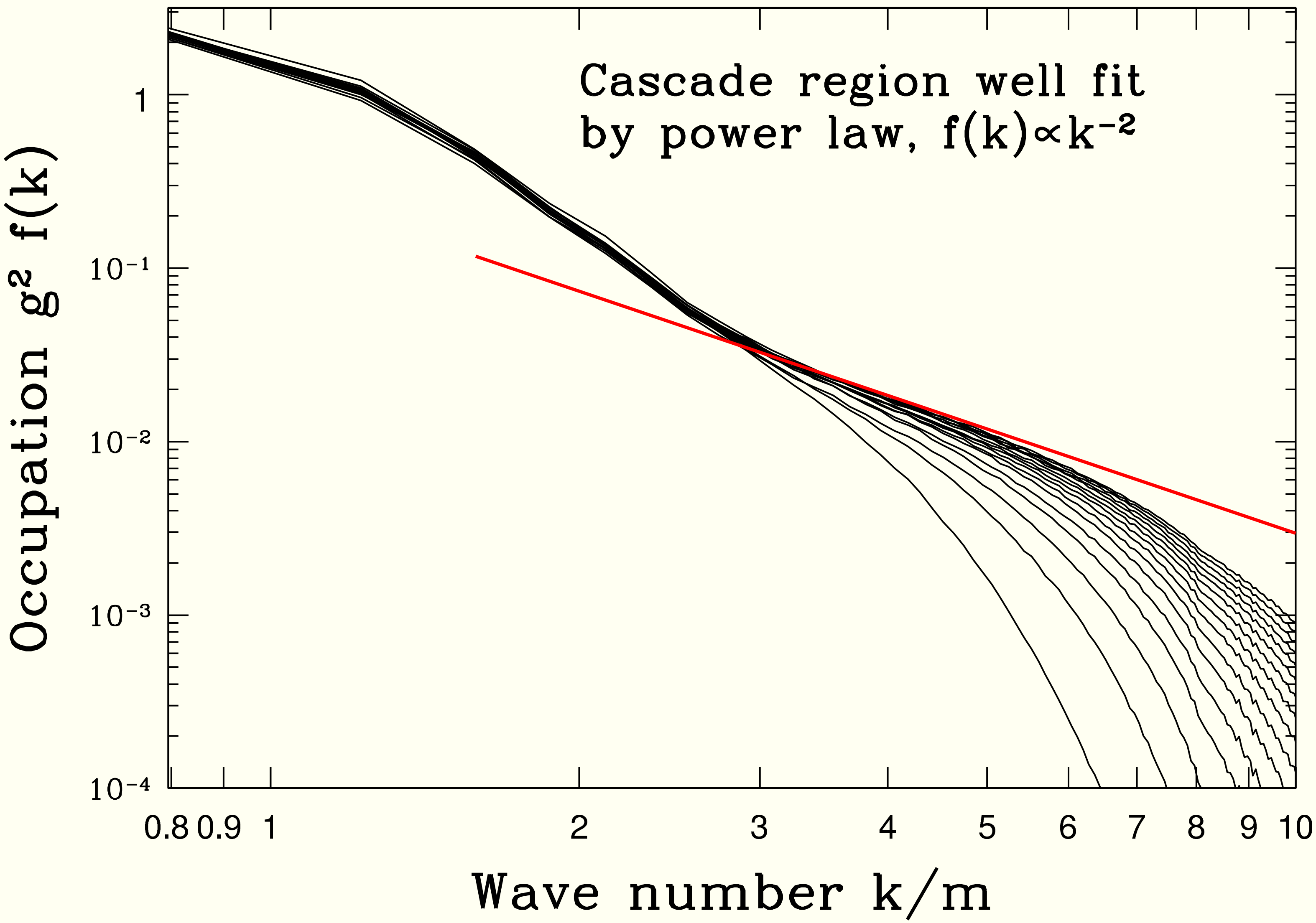
which can be systematically improved by increasing $\mathcal{N}$.

3s $\times$ 3v Hard-loop results – $\xi = 10$



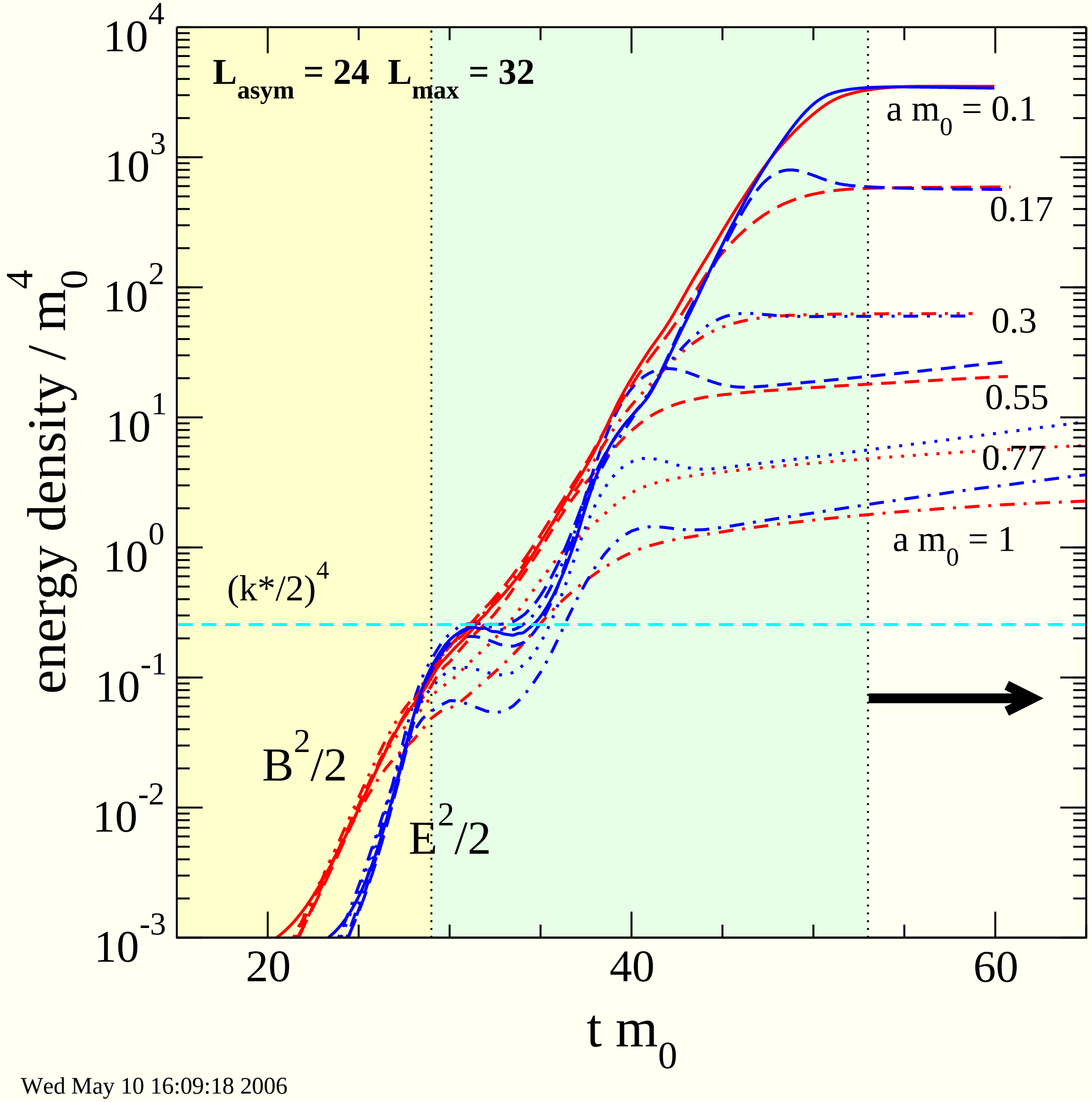
A. Rebhan, P. Romatschke, M. Strickland, hep-ph/0505261

3s $\times$ 3v Hard-loop results – Nonabelian cascade



P. Arnold and G. Moore, hep-ph/0509206; hep-ph/0509226.

3s $\times$ 3v - Larger Anisotropies - $\xi = 100$



D. Bödeker and K. Rummukainen, SEWM06 poster session.

A model of the effect of collisions

We can model the collisional kernel by a Bhatnagar-Gross-Krook (BGK) collision term resulting in a linearized Boltzmann-Vlasov equation of the form

$$[V \cdot D_X, \delta f(p, X)] + g V_\mu F^{\mu\nu} \partial_\nu^{(p)} f(\mathbf{p}) = \mathrm{L}\left(C_{\mathrm{BGK}}[f + \delta f]\right)$$

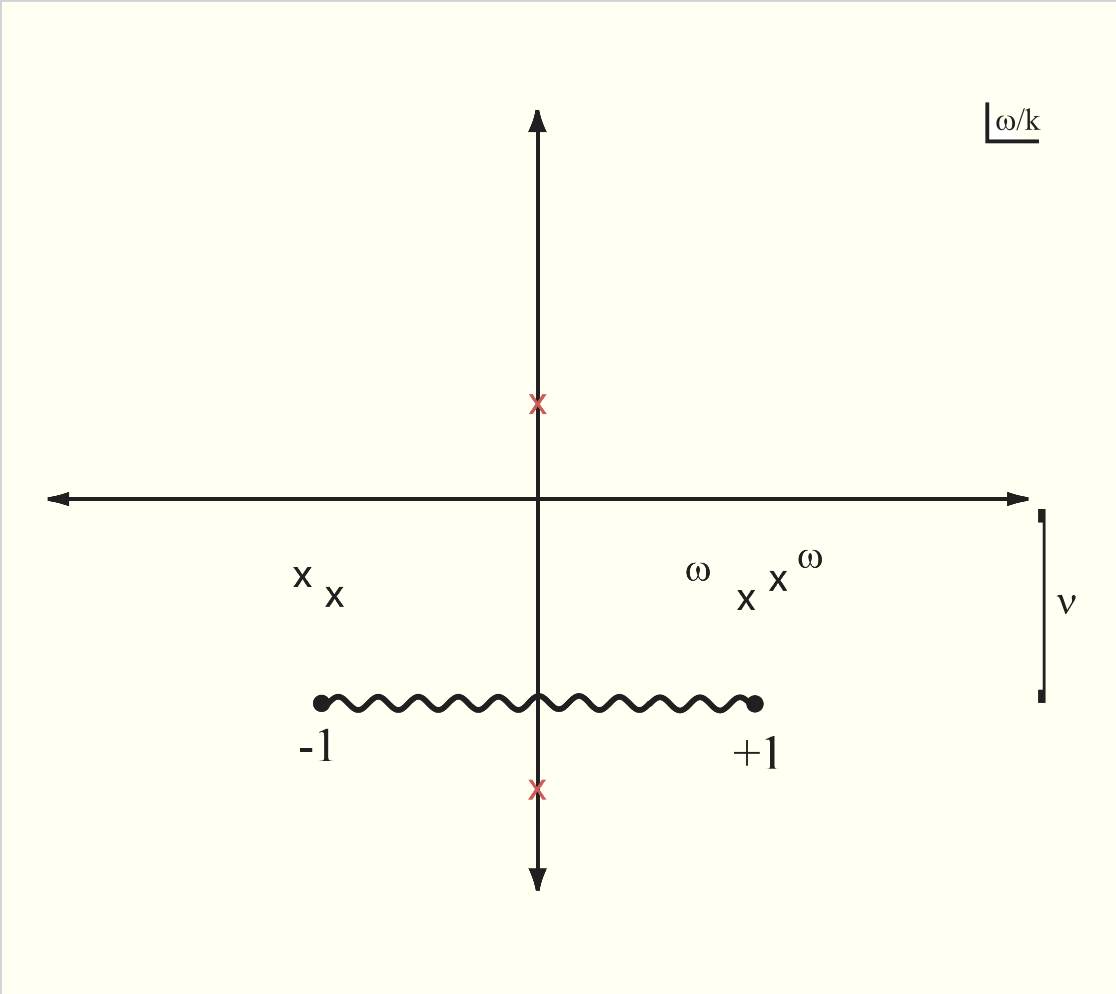
$$C_{\mathrm{BGK}}[f] = -\nu \left(f(p, X) - \frac{N^i(X)}{N^i_{\mathsf{eq}}} f_{\mathsf{eq}}(|\mathbf{p}|) \right)$$

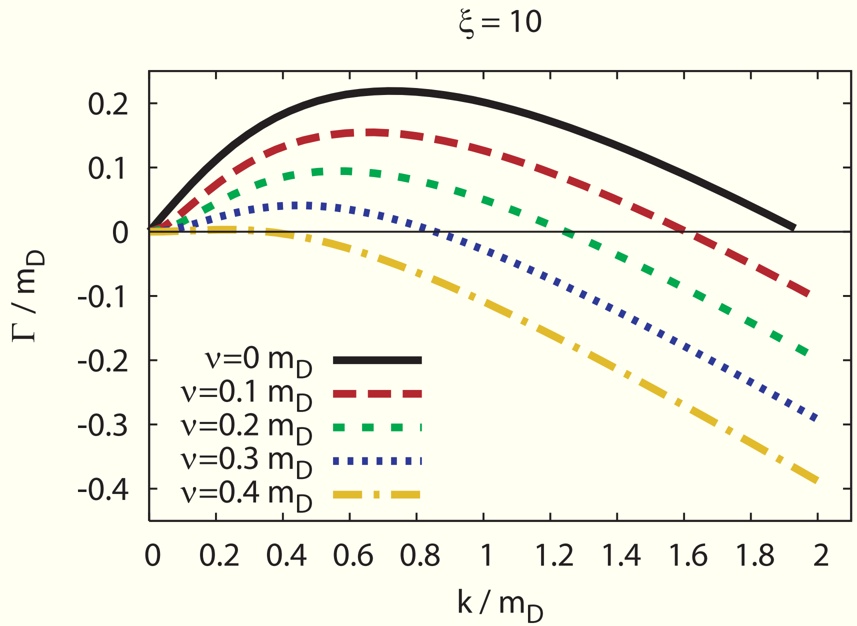
where ν has dimensions of energy and represents the collisional frequency.

For hard, O(1) direction changing interactions $\nu_{\mathrm{hard}}/p_{\mathrm{hard}} \sim g^4 \log g$ and for small-angle ($\theta \sim g$) deflections $\nu_{\mathrm{soft}}/p_{\mathrm{hard}} \sim g^2 \log g$. Assuming the later as an upper bound then $\nu \sim 0.2\, m_D$ when $\alpha_s = 0.3$.

B. Schenke, C. Greiner, M. Thoma, and MS, hep-ph/0603029.

BGK Anisotropic Dispersion Relations





In the limit $\xi \rightarrow \infty$ you can show analytically that there is no instability for

$$\nu > 0.6267\ m_D$$

Colored-Particle-in-Cell Simulations (CPIC)

Hard-loop approximation strictly only applies when there is a large scale separation and weak-field limit ($A \ll p_{\rm hard}/g$).

What happens when one relaxes these assumptions? Let's go back to the transport equations and try to solve without linearization. Recall the Vlasov equation

$$p^\mu[\partial_\mu - g q^a F^a_{\mu\nu}\partial^\nu_p - g f_{abc} A^b_\mu q^c \partial_{q^a}] f(x,p,q) = 0$$

The Vlasov equation is coupled self-consistently to the Yang-Mills equation for the soft gluon fields,

$$D_\mu F^{\mu\nu} = J^\nu = g \int \frac{d^3p}{(2\pi)^3} dq\, q\, v^\nu f(t,\boldsymbol{x},\boldsymbol{p},q)$$

CPIC - Wong-Yang-Mills equations

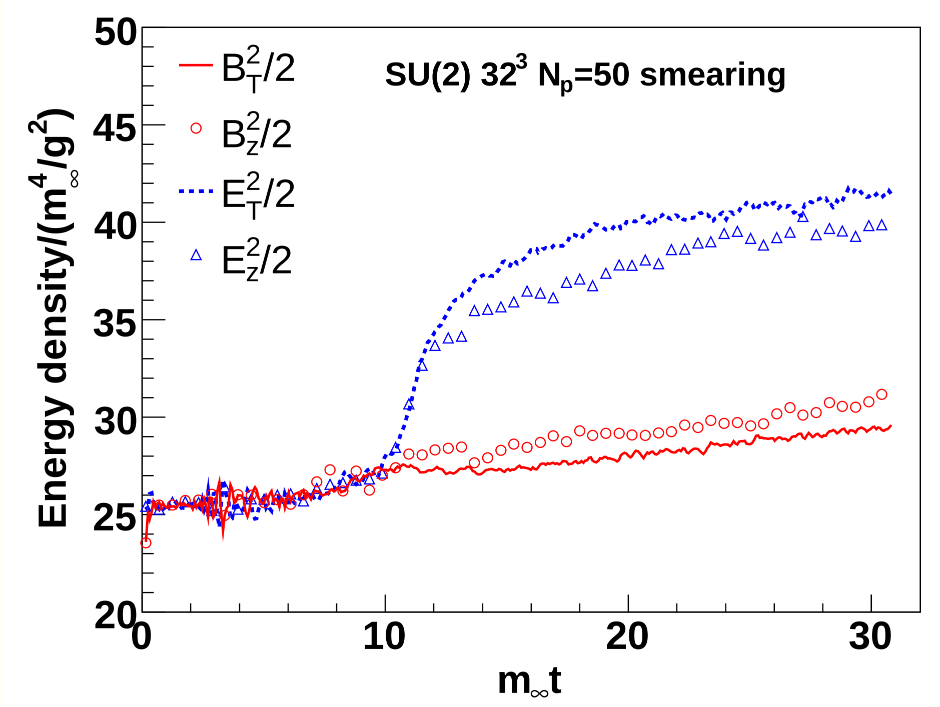
Can be solved numerically by replacing the continuous single-particle distribution $f(\boldsymbol{x}, \boldsymbol{p}, q)$ by a large number of test particles:

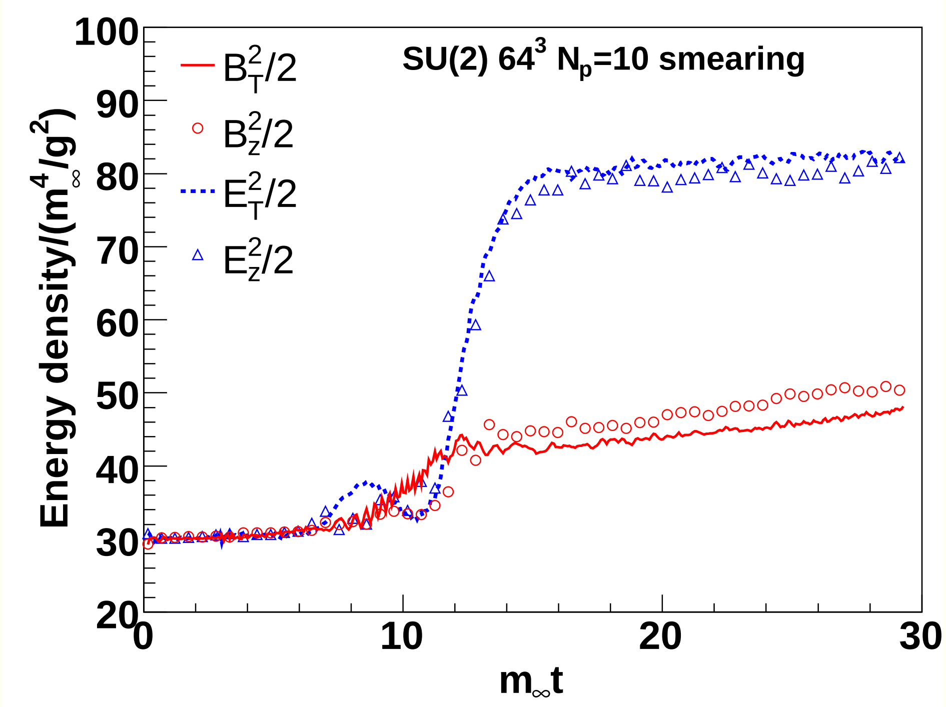
$$f(\boldsymbol{x}, \boldsymbol{p}, q) = \frac{1}{N_{\text{test}}} \sum_i \delta(\boldsymbol{x} - \boldsymbol{x}_i(t)) \, (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{p}_i(t)) \, \delta(q^a - q_i^a(t))$$

where $\boldsymbol{x}_i(t)$, $\boldsymbol{p}_i(t)$ and $q_i^a(t)$ are the coordinates, momentum, and charge of an individual test particle.

$$\begin{aligned}
\frac{d\boldsymbol{x}_i}{dt} &= \boldsymbol{v}_i \\
\frac{d\boldsymbol{p}_i}{dt} &= g \, q_i^a \, (\boldsymbol{E}^a + \boldsymbol{v}_i \times \boldsymbol{B}^a) \\
\frac{d\boldsymbol{q}_i}{dt} &= ig \, v_i^\mu \, [A_\mu, \boldsymbol{q}_i] \\
J^{a\,\nu} &= \frac{g}{N_{\text{test}}} \sum_i q_i^a \, v^\nu \, \delta(\boldsymbol{x} - \boldsymbol{x}_i(t))
\end{aligned}$$

CPIC - Results

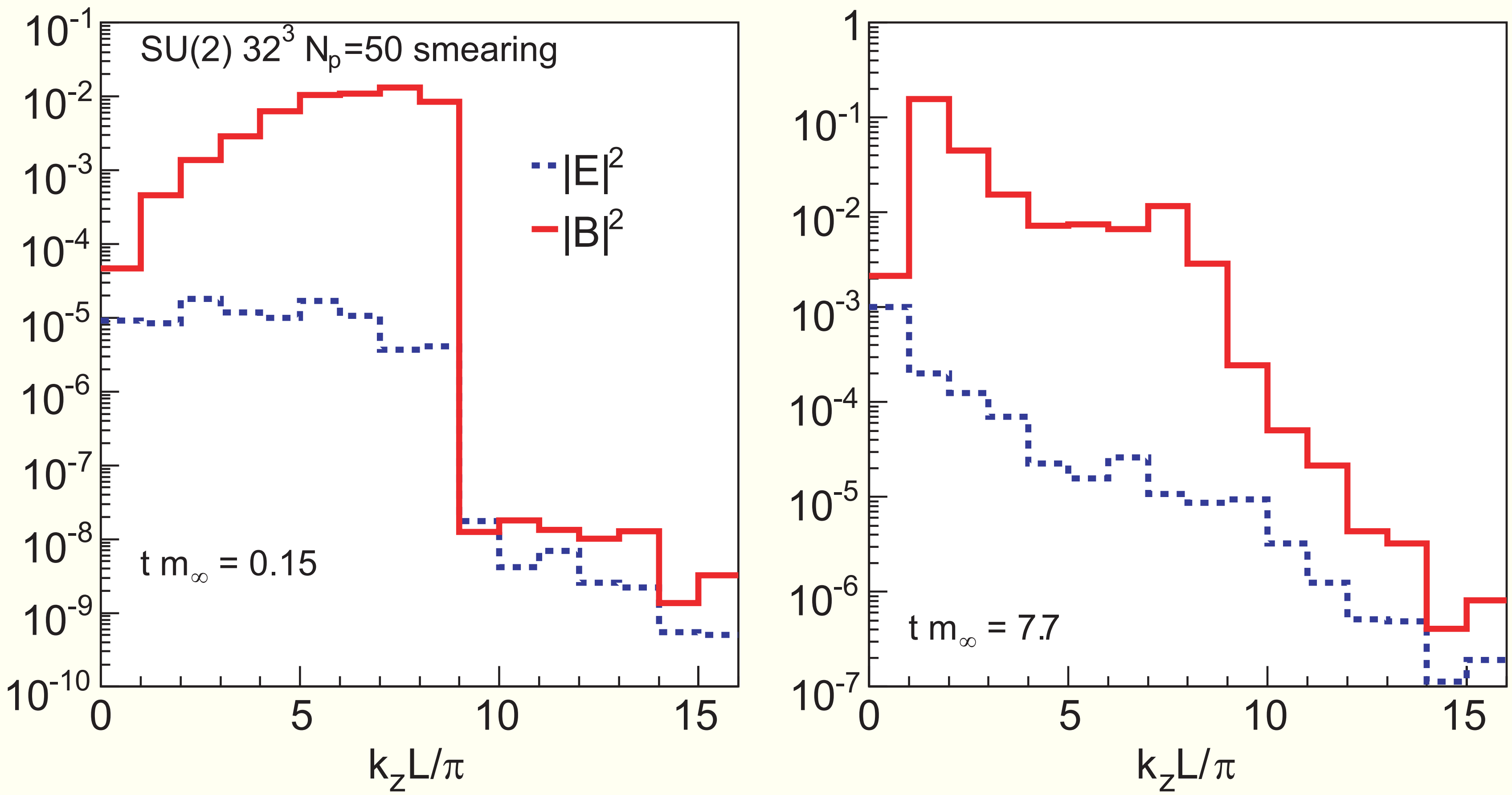




Time evolution of the field energy densities for $SU(2)$ gauge group and anisotropic initial particle momentum distributions. Simulation parameters are $L = 5$ fm, $p_{\rm hard} = 16$ GeV, $g^2\, n_g = 10/\text{fm}^3$, $m_\infty = 0.1$ GeV.

A. Dumitru, Y. Nara, and M. Strickland, hep-ph/0604149

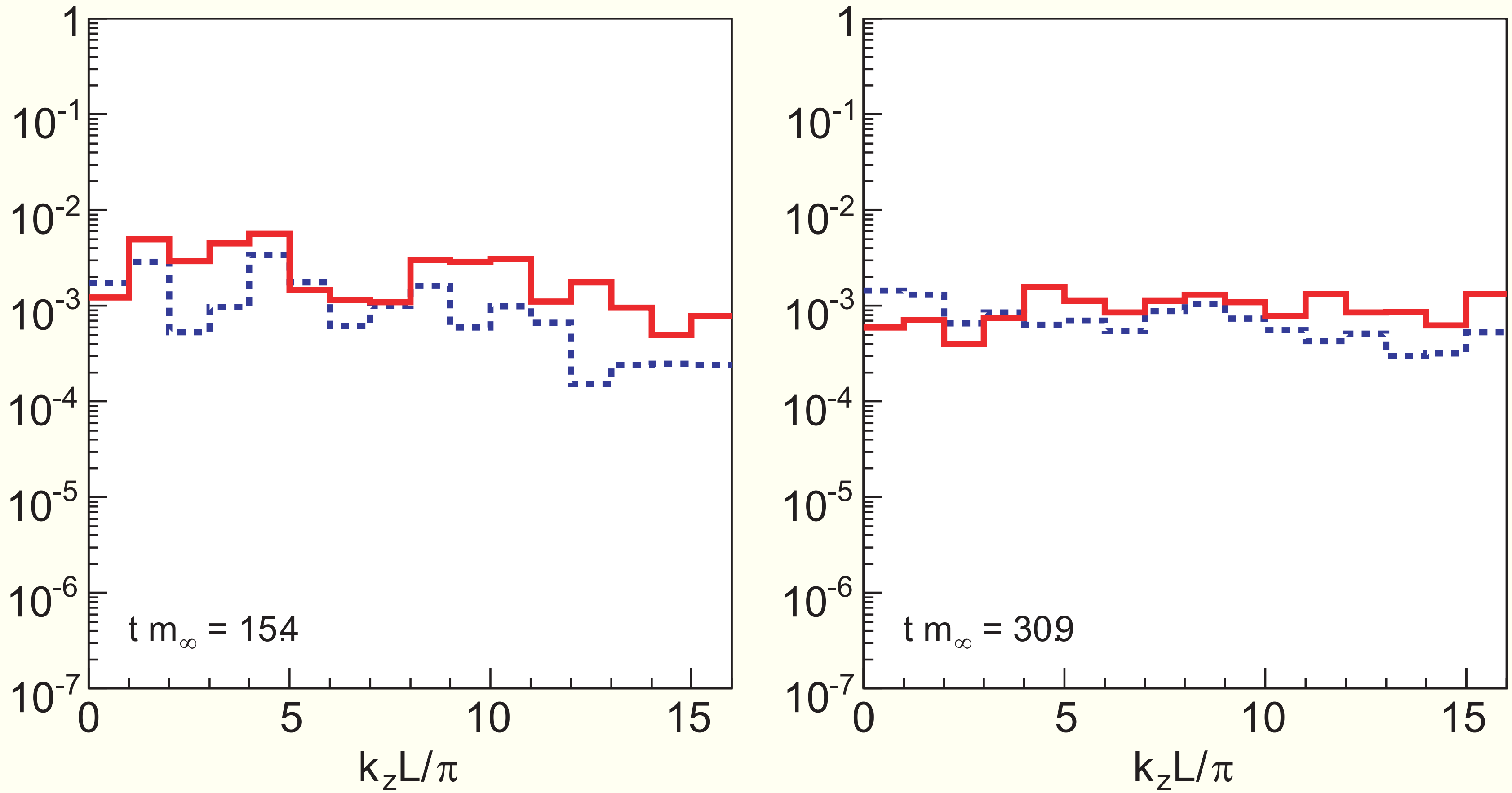
CPIC - Ultraviolet Avalanche



Squares of the Fourier transformed (color-) electric and magnetic fields (in lattice units) at four different times.

A. Dumitru, Y. Nara, and M. Strickland, hep-ph/0604149

CPIC - Ultraviolet Avalanche



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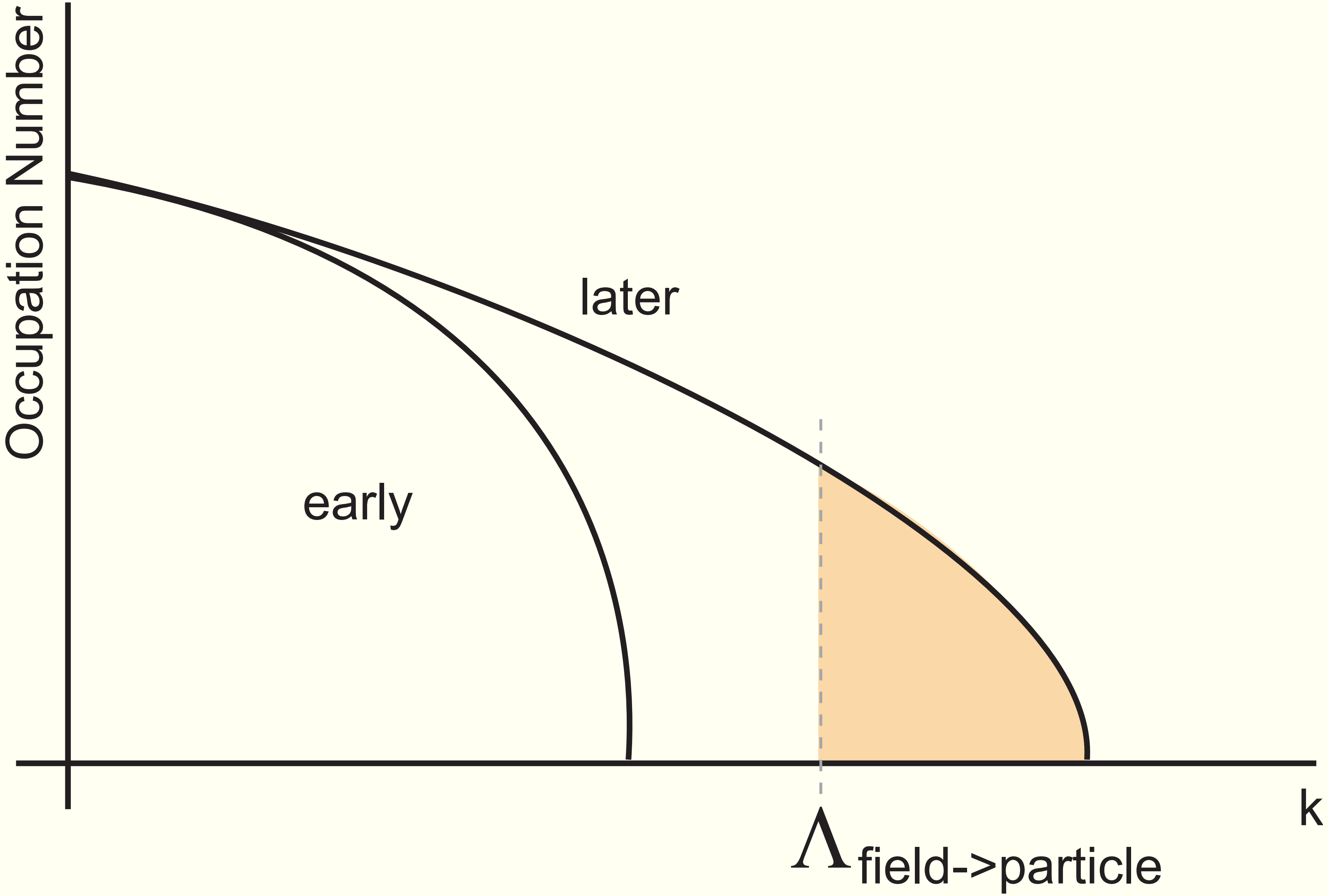
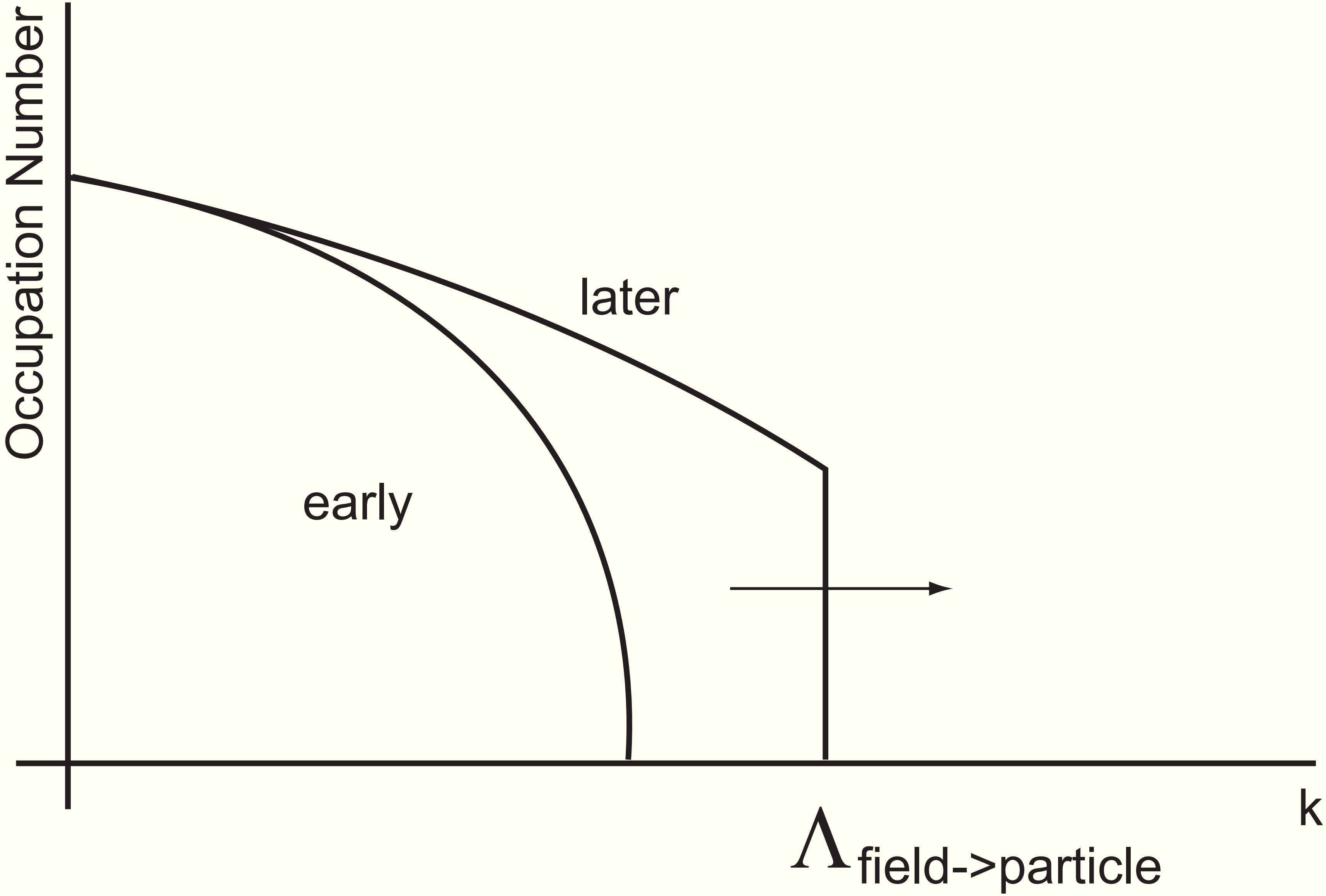
Cycle of isotropization?



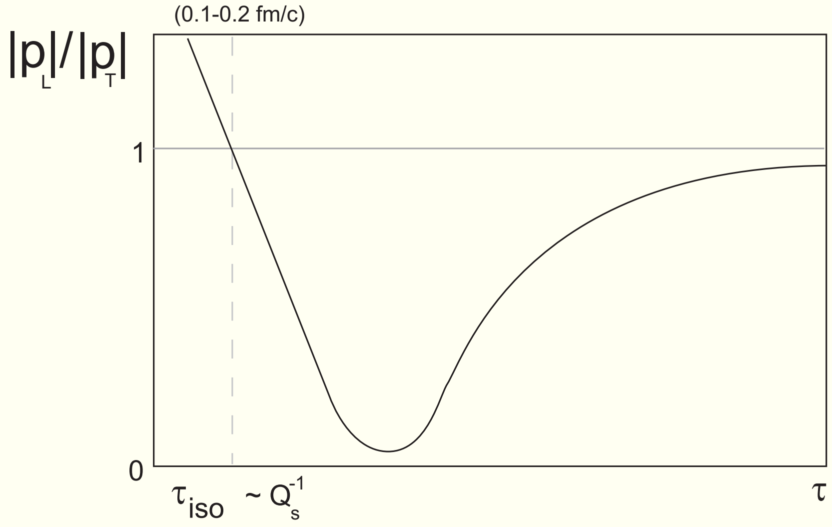
Cycle of isotropization?



Hard Expanding Loops (HEL)

Including expansion at late times one can show

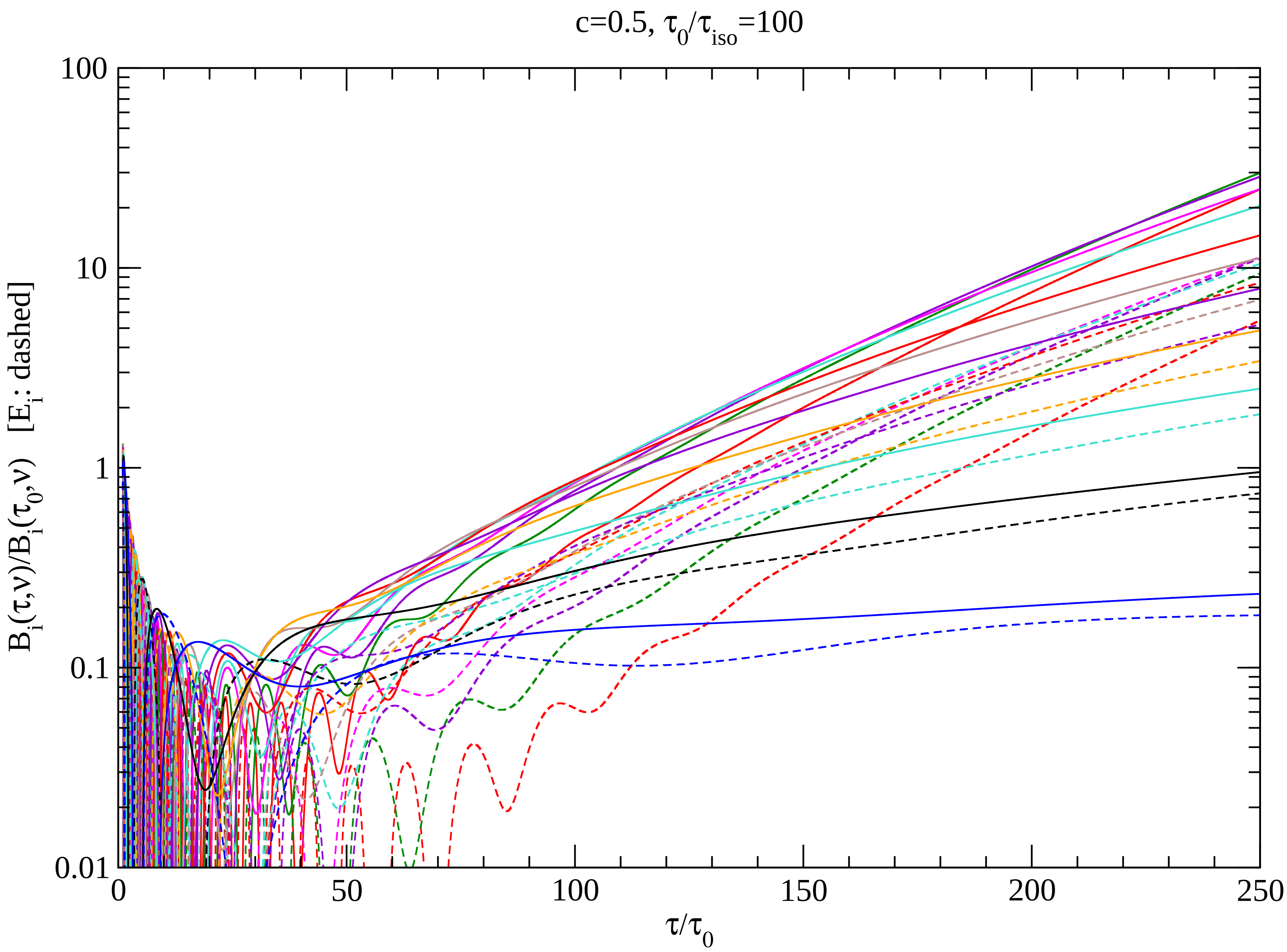
$$
\begin{aligned}
m_\infty(\tau) &\propto \sqrt{\frac{Q_s}{\tau}} \\
A(\tau) &\propto e^{\int_{\tau_0}^{\tau} m_\infty(t)dt} \\
&\propto e^{\sqrt{Q_s \tau}}
\end{aligned}
$$



To include the effect of expansion at all times one must solve the Boltzmann-Vlasov equation in an expanding metric using (τ, x^i, η) coordinates (A. Rebhan and P. Romatschke, hep-ph/0605064.):

$$
\begin{aligned}
p \cdot D\, \delta f^a\big|_{p^\mu} &= g p^\alpha F^a_{\alpha\beta} \partial^\beta_{(p)} f_0(\mathbf{p}_\perp, p_\eta) \\
\frac{1}{\tau} D_\alpha \left(\tau F^{\alpha\beta}\right) &= j^\beta = \frac{g}{2} \int \frac{d^2 p_\perp dy}{(2\pi)^3} p^\beta \delta f
\end{aligned}
$$

Magnetic and electric field strength for each Fourier mode in rapidity



A. Rebhan and P. Romatschke, hep-ph/0605064.

Other recent related works of interest

- “The Unstable Glasma”, Paul Romatschke, Raju Venugopalan, hep-ph/0605045.
- “Anomalous Viscosity of an Expanding Quark-Gluon Plasma”, M. Asakawa, S.A. Bass, B. Müller, hep-ph/0603092
- Your name here!

Conclusions

- Anisotropic plasmas are qualitatively different than isotropic ones. An entirely new phenomena associated with unstable modes appears.
- For relatively weak anisotropies 3 space $\times$ 3 velocity real-time lattice simulations indicate that for non-abelian plasmas the soft unstable modes “saturate” and the growth then becomes power-law rather than exponential.
- However, for larger anisotropies it appears that exponential field growth can continue simliar to an abelian plasma.
- Addition of collisions slows down growth of instabilities but for realistic collisional frequencies instabilities are still present.
- Going beyond the hard-loop approximation by numerically solving the Wong-Yang-Mills equations (CPIC) also shows rapid field growth (but electric fields???) and an “ultraviolet avalanche” accompanied with saturation.